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LETTER TO THE EDITOR

Random resistor networks as a theory of interacting Bose and Fermi fields: I. Effective medium treatment of percolation

J P Carton

Service de Physique du Solide et de Résonance Magnétique, CEN Saclay, 91191 Gif-sur-Yvette, Cedex, France

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Abstract. The problem of random resistor networks is formulated with the help of functional integrals over commuting and anticommuting variables and leads to interacting Bose and Fermi fields. In this letter the effective medium approximation is derived following a variational principle.

In this letter a new approach to percolation in random resistor networks is presented. So far the existing methods for dealing with this problem up to renormalisation treatment are:

(i) equivalence with a random s-states Potts model with $s \to 0$ (Dasgupta *et al* 1978);

(ii) Stephen's approach (Stephen 1978).

They turn out to be quite intricate and rapidly difficult to handle beyond simple cases. From this point of view the present line seems to be more promising. In what follows the formalism is set up and it will be shown how this method allows us to derive an effective medium theory (EMT) as a 'classical' variational theory. It is based on the use of integrals over commuting and anticommuting variables which recently gave some results for disordered systems (Bohr and Efetov 1982, de Dominicis *et al* 1980).

Consider a Bravais lattice with z nearest neighbours, each bond $(i\text{-}j)$ bearing a resistor of conductance σ_{ij}. Kirchhoff's law at node i gives

$$\sum_j \sigma_{ij}(V_i - V_j) = -I_i = \sum_j B_{ij} V_j$$

for the current flowing from the lattice. The relevant quantity for the study of the macroscopic conductance is the voltage response function to a current source between r and r' (see e.g. Lubensky 1978)

$$G(r, r') = \langle r'|B^{-1}|r\rangle.$$

From elementary linear algebra

$$G(r, r') = (\det B)^{-1}\langle r|\mathrm{com}\, B|r'\rangle = (\det B)^{-1}\det \mathscr{B}(r, r')$$

where the matrix $\mathscr{B}(r, r')$ is obtained from B by suppressing line r and column r'.

$(\det B)^{-1}$ is commonly written as a gaussian integral over classical (commuting) fields:

$$(\det B)^{-1} = \int \prod_i \frac{(\mathrm{d}\bar{\varphi}_i \mathrm{d}\varphi_i)}{2\pi \mathrm{i}} \exp - \sum_{ij} \bar{\varphi}_i B_{ij} \varphi_j.$$

Let us introduce a Grassman algebra with $2N$ generators η_i and $\bar{\eta}_i$ which completely anticommute,

$$\eta_i \bar{\eta}_j + \bar{\eta}_j \eta_i = 0, \qquad \eta_i \eta_j + \eta_j \eta_i = 0,$$

and an integration symbol over $\{\eta_i\}$ conventionally defined by

$$\int \mathrm{d}\eta \ 1 = \int \mathrm{d}\bar{\eta} \ 1 = 0, \qquad \int \mathrm{d}\eta \ \eta = \int \mathrm{d}\bar{\eta} \ \bar{\eta} = 1$$

(for more details see Berezin (1966)). Then the following representation holds:

$$\det B = \int \prod_i (\mathrm{d}\bar{\eta}_i \mathrm{d}\eta_i) \exp - \sum_{ij} \bar{\eta}_i B_{ij} \eta_j \tag{1}$$

((1) is readily obtained through a diagonalisation of B).

Moreover, since $\eta_i^2 = 0$ one sees that if a linear term $\bar{\eta}_r$ is introduced in integral (1) the contribution of $\int \mathrm{d}\bar{\eta}_r$ is -1, so that

$$\int \prod_i (\mathrm{d}\bar{\eta}_i \ \mathrm{d}\eta_i) \bar{\eta}_r \eta_{r'} \exp - \sum \bar{\eta}_i B_{ij} \eta_j = -\det \mathscr{B}(r, r').$$

Consequently G appears as the correlation of a Fermi field in a theory involving both Fermi and Bose fields:

$$-G(r, r') = \int \prod_i \left(\frac{1}{2\pi \mathrm{i}} \mathrm{d}\bar{\varphi}_i \ \mathrm{d}\varphi_i \ \mathrm{d}\bar{\eta}_i \ \mathrm{d}\eta_i \right) \bar{\eta}_r \eta_{r'} \exp - \sum_{ij} B_{ij} (\bar{\varphi}_i \varphi_j + \bar{\eta}_i \eta_j). \tag{2}$$

In the present case

$$\sum_{ij} B_{ij} (\bar{\varphi}_i \varphi_j + \bar{\eta}_i \eta_j) = \sum_{\{i,j\}} \sigma_{ij} [(\bar{\varphi}_i - \bar{\varphi}_j)(\varphi_i - \varphi_j) + (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j)]$$

where summation is extended to bonds between nearest neighbours.

By construction the probability law in (2) is automatically normalised to unity, which allows an averaging procedure as usual in disorder problems. For example, assume that $\sigma_{ij} = \sigma$ with probability p and $\sigma_{ij} = 0$ with probability $1-p$; then

$$\overline{-G(r, r')} = \int \prod_i \left(\frac{1}{2\pi \mathrm{i}} \mathrm{d}\bar{\varphi}_i \ \mathrm{d}\varphi_i \ \mathrm{d}\bar{\eta}_i \ \mathrm{d}\eta_i \right) \bar{\eta}_r \eta_{r'} \ \mathrm{e}^{-\mathscr{L}}$$

where the effective (including interactions) Lagrangian is

$$\mathscr{L} = -\sum \ln [p \exp(-\sigma \phi_{ij}) + 1 - p]$$

setting

$$\phi_{ij} = (\bar{\varphi}_i - \bar{\varphi}_j)(\varphi_i - \varphi_j) + (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j).$$

Note that this theory is massless and that one expects for the quasi-particle spectrum of the Fermi field a form

$$\varepsilon(k) = k^2(a + bk^2 + \ldots)$$

where a is related to the macroscopic conductivity Σ (Lubensky 1978) (in the homogeneous case $G(k) \sim 1/\sigma k^2$). The percolation threshold should thus be characterised by the vanishing of a.

For some purpose it can be convenient to separate $\mathscr{L}$ into free parts $\mathscr{L}_B$ and $\mathscr{L}_F$ respectively for φ and η and an interaction Lagrangian $\mathscr{L}_I$; using anticommutation rules one finds

$$\mathscr{L}_B = -\sum_{\{i,j\}} \ln [p \exp(-\sigma\Delta_{ij}) + 1 - p], \qquad \mathscr{L}_F = p\sigma \sum_{\{i,j\}} (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j),$$

$$\mathscr{L}_I = p(p-1)\sigma \sum_{\{i,j\}} [p \exp(-\sigma\Delta_{ij}) + 1 - p]^{-1}[\exp(-\sigma\Delta_{ij}) - 1](\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j)$$

where

$$\Delta_{ij} = (\bar{\varphi}_i - \bar{\varphi}_j)(\varphi_i - \varphi_j).$$

The simplest approximation—namely neglecting $\mathscr{L}_I$—yields $\Sigma \sim p\sigma$ and leaves percolation phenomena aside. One step further is for example to construct a self-consistent theory for Σ. This will be achieved through a variational method with a trial Lagrangian chosen as representing an effective medium:

$$\mathscr{L}_0 = \sum_{\{i,j\}} \sigma_0 |\eta_i - \eta_j|^2 + \sum_{\{i,j\}} \sigma_0' |\varphi_i - \varphi_j|^2.$$

The variational free energy $\mathscr{F}$ as a function of the trial parameters σ_0 and σ_0' is

$$\mathscr{F}(\sigma_0, \sigma_0') = F_0 + \langle \mathscr{L} - \mathscr{L}_0 \rangle_0 = -\ln \mathrm{Tr}\, \mathrm{e}^{-\mathscr{L}_0} + [\mathrm{Tr}(\mathscr{L} - \mathscr{L}_0)\mathrm{e}^{-\mathscr{L}_0}](\mathrm{Tr}\, \mathrm{e}^{-\mathscr{L}_0})^{-1}.$$

One readily has

$$F_0/N = -\ln(\sigma_0/\sigma_0') + \text{constant}, \qquad \langle \mathscr{L}_0 \rangle = 0,$$

which explains why $\sigma_0 = \sigma_0'$ cannot be imposed from the beginning.

To calculate $\langle \mathscr{L} \rangle_0$ it is convenient to introduce a Fourier representation

$$\ln(1 + x\, \mathrm{e}^{-\sigma\phi}) = \int \mathrm{d}\lambda\ \Lambda(\lambda)\, \mathrm{e}^{\mathrm{i}\lambda\phi} \tag{3}$$

where $x = p/1 - p$. Then

$$\langle \mathscr{L} \rangle_0 = -\sum_{\{ij\}} \int \mathrm{d}\lambda\ \Lambda(\lambda) \langle \exp \mathrm{i}\lambda\, \phi_{ij} \rangle_0, \tag{4}$$

$$\langle \exp \mathrm{i}\lambda\, \phi_{ij} \rangle_0 = \langle \exp \mathrm{i}\lambda\, |\varphi_i - \varphi_j|^2 \rangle_0 \langle \exp \mathrm{i}\lambda\, |\eta_i - \eta_j|^2 \rangle_0.$$

From the properties of gaussian laws it is straightforward to evaluate

$$\langle \exp \mathrm{i}\lambda\, |\varphi_i - \varphi_j|^2 \rangle_0 = (1 - 2\mathrm{i}\lambda/z\sigma_0')^{-1} \tag{5}$$

whereas

$$\langle \exp \mathrm{i}\lambda\, (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j) \rangle_0 = 1 + \mathrm{i}\lambda \langle (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j) \rangle_0 = 1 - 2\mathrm{i}\lambda/z\sigma_0 \tag{6}$$

since $\{\eta_i, \bar{\eta}_j\}$ is an anticommuting set.

From (3), (4), (5), (6) one obtains the minimisation equations

$$N^{-1} \frac{\partial \mathscr{F}}{\partial \sigma_0} = -\frac{1}{\sigma_0^2}\left(\sigma_0 + \int \mathrm{d}\lambda\ \Lambda(\lambda) \frac{\mathrm{i}\lambda}{1 - 2\mathrm{i}\lambda/z\sigma_0'}\right) = 0,$$

$$N^{-1}\frac{\partial \mathscr{F}}{\partial \sigma_0'}=\frac{1}{\sigma_0'^2}\left(\sigma_0'+\int \mathrm{d}\lambda\ \Lambda(\lambda)(\mathrm{i}\lambda)\frac{1-2\mathrm{i}\lambda/z\sigma_0}{(1-2\mathrm{i}\lambda/z\sigma_0')^2}\right)=0.$$

Looking for solutions $\sigma_0=\sigma_0'$, continuous for $\sigma_0\neq 0$, one simply has

$$\sigma_0-\tfrac{1}{2}z\sigma_0\int \mathrm{d}\lambda\ \Lambda(\lambda)+\tfrac{1}{2}z\sigma_0\int \mathrm{d}\lambda\ \Lambda(\lambda)\frac{1}{1-2\mathrm{i}\lambda/z\sigma_0}=0, \tag{7}$$

$$\int \mathrm{d}\lambda\ \Lambda(\lambda)=\ln(1+x)=-\ln(1-p).$$

The linear term in σ_0 in (7), i.e. $1+\frac{1}{2}z\ln(1-p)$, vanishes at $p_c=1-e^{-2/z}$. Close to p_c, σ_0 is small

$$\int \mathrm{d}\lambda\ \Lambda(\lambda)\frac{1}{1-2\mathrm{i}\lambda/z\sigma_0}=-\frac{z\sigma_0}{2\mathrm{i}}\int \mathrm{d}\lambda\ \Lambda(\lambda)\frac{1}{\lambda+\frac{1}{2}\mathrm{i}z\sigma_0}$$

$$\sim \pi z\sigma_0\int_0^\infty \mathrm{d}\phi\ \ln(1+x\,e^{-\sigma\phi})=\pi z(\sigma_0/\sigma)I(x)$$

and

$$\sigma_0\sim -2\frac{1+\frac{1}{2}z\ln(1-p)}{\pi z^2 I(x)}\sigma.$$

As in EMT (Kirkpatrick 1973), σ_0 would be negative for $p<p_c$, which is unphysical, so that $\sigma_0=0$ for $p<p_c$, and on the other hand

$$\sigma_0\sim p-p_c \qquad \text{for } p>p_c.$$

EMT gives $p_c=2/z$ which is the same as our $p_c=1-e^{-2/z}$ when $z\to\infty$. For $p=1$, $\Lambda(\lambda)=-\mathrm{i}\sigma\delta'(\lambda)$ and the solution of (7) is immediately $\sigma_0=\sigma$.

The instability revealed by the existence of a solution $\sigma_0<0$ suggests the onset of a broken symmetry phase when $p<p_c$. This will be shown in the following. Write

$$\varphi_i=\varphi_i^0+\delta\varphi_i, \qquad \eta_i=\eta_i^0+\delta\eta_i,$$

where $\{\varphi^0\}$ and $\{\eta^0\}$ are values of the fields imposed by external constraints. Integrating out the fluctuations $\{\delta\varphi\}$ and $\{\delta\eta\}$ through a saddle point method gives the free energy (or vertex generating functional) as a function of $\{\varphi^0\}$ and $\{\eta^0\}$. Choosing

$$(\bar\varphi_i^0-\bar\varphi_j^0)(\varphi_i^0-\varphi_j^0)=\Delta, \qquad (\bar\eta_i^0-\bar\eta_j^0)(\eta_i^0-\eta_j^0)=\Xi,$$

one may expand $\mathscr{L}$ as

$$\mathscr{L}(\varphi,\eta)=\mathscr{L}(\varphi^0,\eta^0)+\sum_{\{i,j\}}Q_{\alpha\beta}(\Delta,\Xi)\bar u_\alpha \bar u_\beta$$

where $u=(\delta\bar\varphi_i-\delta\bar\varphi_j,\delta\varphi_i-\delta\varphi_j,\delta\bar\eta_i-\delta\bar\eta_j,\delta\eta_i-\delta\eta_j)$. After cumbersome integration over $\delta\varphi$ and $\delta\eta$ one is left with an expression for the free energy $\Gamma(\Delta,\Xi)$ in the so-called 'one-loop approximation':

$$\Gamma(\Delta,\Xi)=-\frac{z}{2}l(\Delta)+\tfrac{1}{2}\ln\left(1+2\frac{l''(\Delta)}{l'(\Delta)}\Delta\right)+\Xi\frac{\mathrm{d}}{\mathrm{d}\Delta}\left[-\frac{z}{2}l(\Delta)+\tfrac{1}{2}\ln\left(1+2\frac{l''}{l'}\Delta\right)\right] \tag{8}$$

where $l(\Delta)=\ln(p\,e^{-\sigma\Delta}+1-p)$.

The form of (8) is quite general, as a consequence of the required property

$$\langle|\varphi_i-\varphi_j|^2+|\eta_i-\eta_j|^2\rangle=0.$$

It yields

$$\Sigma=(\lim_{k\to 0} k^2\bar{G}(k))^{-1}=\frac{2}{z}\frac{\mathrm{d}}{\mathrm{d}\Delta}\Gamma(\Delta,0)\bigg|_{\Delta=\Delta_0}$$

where Δ_0 is given by

$$\partial\Gamma/\partial\varphi_i^0=(\mathrm{d}/\mathrm{d}\Delta)\Gamma(\Delta_0,0)(\bar{\varphi}_i^0-\bar{\varphi}_j^0)=0.$$

Expanding

$$\Gamma(\Delta,0)=\tfrac{1}{2}zp\sigma\Delta-(1-p)\sigma\Delta+\mathrm{O}(\Delta^2)$$

one sees that:

(i) when $p>p_c=(1+z/2)^{-1}$, $\varphi_i^0-\varphi_j^0=0$ and $\Sigma=\sigma(1+2/z)(p-p_c)$;

(ii) when $p<p_c$ the symmetry is spontaneously broken so as to have $\Delta_0\neq 0$ given by $(\mathrm{d}/\mathrm{d}\Delta)\Gamma(\Delta_0,0)=0$ which in turn implies $\Sigma=0$ for all $p<p_c$.

Another peculiar feature is the following: contrary to standard critical phenomena the space dimension d does not explicitly appear in Γ to order one loop; this does not allow us to control the validity of the approximation through a Ginzburg criterion. Probably it is never exact since it gives for the conductivity the exponent $t=1$ whereas the classical (Landau) value is known to be $t=3$. In that sense the above treatment does not deserve the name of 'mean field approximation'.

In conclusion this method avoids replication, its traps (known and hidden) and any kind of analytic continuation. The next step is naturally the renormalisation group for which the known tools of quantum field theory are available. On the other hand, more complicated situations can be treated in the same fashion: spatially varying distribution (e.g. surface effects), crossover around the percolation point for $\sigma_{ij}=\sigma_>$ or $\sigma_<$, percolation with shorts or diodes.

References

Berezin F A 1966 *The method of second quantization* (New York: Academic)
Bohr T and Efetov K G 1982 *J. Phys. C: Solid State Phys.* **15** L249
Dasgupta C, Harris A B and Lubensky T C 1978 *Phys. Rev.* **B17** 1375
de Dominicis C, Gabay M, Garel T and Orland H 1980 *J. Physique* **41** 923
Kirkpatrick S 1973 *Rev. Mod. Phys.* **43** 574
Lubensky T C 1978 in *Ill condensed matter, Les Houches 1978* ed R Balian (Amsterdam: North-Holland)
Stephen M J 1978 *Phys. Rev.* **B17** 4444